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A SEMINAR ON " THEORY OF SRINIVASA RAMANUJAN: A CRITICAL STUDY"

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Venue: DEPARTMENT OF MATHEMATICS ,PATTAMUNDAI COLLEGE, PATTAMUNDAI.

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REPORT

A seminar was organised by Department of Mathematics ,Pattamundai College, Pattamundai on 14.02.2019 on the topic "THEORY OF SRINIVASA RAMANUJAN : A CRITICAL STUDY".Dr Ratikanta Kamila , Former Associate Professor of Mathematics, Kendrapara Autonomous College was resource person for the seminar.In this seminar most of the students were present .Prof Ramesh Chandra Sahoo , Principal, chaired the meeting. Prof Arabinda Pandab, Head of the Department gave a key note address of the topic and welcomed the guests on the podium and the participants. The meeting was ended with vote of thanks by Dr Nirmal Kumar Sahoo, another faculty member.

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THEORY OF SRINIVASA RAMANUJAN : A CRITICAL STUDY

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Abstract

The purpose of this paper is to introduce some of the contributions of Srinivasa Ramanujan to number theory. The following topics are covered in this paper: Magic squares, Theory of partitions, Ramanujan's contribution to the concept of highly composite numbers, Expressions for π, Diophantine equations, Ramanujan's number, a symmetric equation and Ramanujan's equation.

Key words: The partition function $p(n)$-Congruences-Highly composite numbers-Diophantine equations.

1. Introduction

While writing the biography of Srinivasa Ramanujan in 1991, Robert Kanigel [6] gave the title to his book as 'The man who knew infinity.' J. L. Littlewood described Ramanujan with the words 'Every positive integer is one of his personal friends.' Their expressions amply describe the heights in the realm of numbers to which Ramanujan rose. The purpose of this paper is to bring out some of the contributions made by Ramanujan to Number theory.

2. Magic squares

As a young student, Srinivasa Ramanujan was interested in magic squares. The simplest magic square problem is to fill up the cells in a square with 3 rows and 3 columns with the numbers $1, 2, 3, \cdots, 9$ such that each row sum = each column sum = each diagonal sum. One can find the following solution:

4	9	2
3	5	7
8	1	6

In this magic square, each row sum = each column sum = each diagonal sum = 15. This magic square requires an understanding of how 15 can be written as a sum of 3 non-repeated numbers.

One can think of magic squares of types $4 \times 4, 5 \times 5, \cdots$. Ramanujan gave the following formula for a general magic square of type 3×3 :

C+Q	A+P	B+R
A+R	B+Q	C+P
B+P	C+R	A+Q

where A, B, C (respectively P, Q, R) are integers in arithmetic progression. The following formula was also provided by him.

2Q+R	2P+2R	P+Q
2P	P+Q+R	2Q+2R
P+Q+2R	2Q	2P+R

3. Theory of partitions

The magic squares form the nucleus of the theory of partitions developed by Srinivasa Ramanujan. His fascination for magic squares led him in his later life to work on this theory. Let us consider the partitions of a natural number. Let $p(n)$ denote the partition function n, defined as the number of ways of expressing n as a sum of natural numbers $\leq n$.

For example, 1 has the partition 1; 2 has the partitions 2, 1+1; 3 has the partitions 3, 2+1, 1+1+1, and so on. As n increases, $p(n)$ becomes larger and larger. For example, 6 has the partitions 6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1.

The following table provides the values of $p(n)$ for $n = 1, 2, \cdots, 20$.

Table 1

n	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	2	3	5	7	11	15	22	30	42
n	11	12	13	14	15	16	17	18	19	20
$p(n)$	56	77	101	135	176	231	297	385	490	627

Denoting the number of partitions of n with parts $\leq m$ by $p_m(n)$, we have the recurrence relation $p_m(n) = p_{m-1}(n) + p_m(n-m)$ $(1 < m \leq n)$. About the partitions of a natural number, G. H. Hardy and E. M. Wright remarked in [5], "in spite of the simplicity of the definition of $p(n)$, not very much is known about its arithmetic properties". Under this back ground, it is worth while to consider the contributions of Ramanujan.

Three papers by Rmanujan on the theory of partitions were published in the years 1919, 1920 and 1921.

Ramanujan's congruences

When m is 0 or a natural number, Ramanujan obtained the following congruences: $p(5m+4) \equiv 0 \pmod 5$, $p(7m+5) \equiv 0 \pmod 7$, $p(11m+6) \equiv 0 \pmod{11}$.

Regarding the contribution of Ramanujan to the theory of partitions, G. H. Hardy and E. M. Wright wrote "The simplest arithmetic properties were found by Ramanujan. Examining Mac Mahon's table of $p(n)$, he was first led to conjecture and then to prove, three striking arithmetic properties associated with the moduli 5, 7 and 11 ". To illustrate the results of Ramanujan, the values of $p(n)$ for $n \equiv 4 \pmod 5$, $5 \pmod 7$, $6 \pmod{11}$ are provided in tables 2, 3, 4, respectively.

Table 2

n	4	9	14	19	24	29	34	39
$p(n)$	5	30	135	490	1575	4565	12310	31185

Table 3

n	5	12	19	26	33	40
$p(n)$	7	77	490	2436	10143	37338

Table 4

n	6	17	28	39
$p(n)$	11	297	3718	31185

4. Highly composite numbers

A natural number n is said to be composite if it has a divisor different from 1 and itself. Ramanujan raised an interesting question: If n is a composite number, what makes it a highly composite one? For this purpose, he considered the number of distinct positive divisors of n denoted by $d(n)$.

Definition: Highly composite number (Srinivasa Ramanujan [10])

A natural number n is a highly composite number if $d(m) < d(n)$ for all $m < n$.

When one considers the primes and composite numbers in $\mathbb{Z}$, 1 is a unit element and 2 is a prime. However, both of them become highly composite numbers as per the definition of Ramanujan. The first few highly composite numbers and the number of their distinct positive divisors are furnished in the following table.

Table 5

n	1	2	4	6	12	24	36	48	60	120	180
$d(n)$	1	2	3	4	6	8	9	10	12	16	18
n	240	360	720	840	1260	1680	2520	5040	7560	10080	
$d(n)$	20	24	30	32	36	40	48	60	64	72	

One of the highly composite numbers calculated by Ramanujan is
6 7 4 6 3 2 8 3 8 8 8 0 0.

This number has 13 digits and its prime factorization is $2^6.3^4.5^2.7^2.11.13.17.19.23$.

Theorem: Form of a highly composite number (Ramanujan [10])

If $n = 2^{a_1}3^{a_2}5^{a_3}\cdots p^{a_p}$ *is a highly composite number, then* $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_p$ *and* $a_p = 1$ *except for* $n = 4$ *and 36.*

It is to be noted that $4 = 2^2$ and $36 = 2^2.3^2$

Theorem: (Ramanujan [10]) *There are an infinite number of highly composite* *numbers.*

Theorem: *Successive highly composite numbers are asymptotically equivalent.*

Theorem: *If $Q(x)$ denotes the number of highly composite numbers $\leq x$, then* $\lim_{n\to\infty} \frac{Q(x)}{log_e x} = \infty$.

Ramanujan [10] published a paper on highly composite numbers in 1915. He was awarded B. A. Degree by research by Cambridge University in 1916 for his dissertation titled 'Highly composite numbers ' and his advisors were G. H. Hardy and J. E. Littlewood.

5. Formulae for π

For the transcendental number π, Ramanujan has given several formulae. Some of them are listed below:

$$\pi = 24\tan^{-1}\frac{1}{8} + 8\tan^{-1}\frac{1}{57} + 4\tan^{-1}\frac{1}{239}$$

$$\pi \approx \frac{63}{15}.\frac{17+15\sqrt{15}}{7+15\sqrt{15}}$$

$$\pi = \cfrac{4}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+5^2+\cdots}}}$$

6. Diophantine equations

Algebraic equations requiring solutions in integers are called Diophantine equations. They are named after the mathematician of Alexandria, Diophantus. For Diophantine equations, one may refer to L. J. Mordell [7]. Let us briefly see the contributions of Srinivasa Ramanujan to Diophantine equations.

6.1 Ramanujan's number

The number 1729 has acquired a special status in mathematics. It is refered to as Ramanujan number. There is a famous anecdote about this number. Ramanujan made a statement to G. H. Hardy that 1729 is the smallest number that can be expressed as a sum of two cubes in two different ways. We have the two expressions $1729 = 9^3 + 10^3$ and $1729 = 1^3 + 12^3$. Thus 1729 is the smallest integral solution of the equation $A^3 + B^3 = C^3 + D^3$.

Ramanujan gave the general solution of this equation as

$(\alpha + \lambda^2\gamma)^3 + (\lambda\beta + \gamma)^3 = (\lambda\alpha + \gamma)^3 + (\beta + \lambda^2\gamma)^3$ where $\alpha^2 + \alpha\beta + \beta^2 = 3\lambda\gamma^2$.

6.2. A variant of Fermat's last theorem

Ramanujan's number has another implication also. Pierre de Fermat (1601-1665) stated the famous conjecture that when $n > 2$, the equation $x^n + y^n = z^n$ has no integral solution except $x = y = z = 0$. This is popularly known as Fermat's last theorem. A special case of the above equation is $x^3 + y^3 = z^3$. Euler proved that there is no non-trivial integral solution for this equation. A variant of this special case is the equation $x^3 + y^3 = z^3 + 1$. Ramanujan's number provides the solution $9^3 + 10^3 = 12^3 + 1$ to the above Diophantine equation.

6.3. The Diophantine equation $X^3 + Y^3 + Z^3 = W^3$

Ramanujan found out the following parametric solution to the Diophantine equation $X^3 + Y^3 + Z^3 = W^3$:

$X = 3a^2 + 5ab - 5b^2$

$Y = 4a^2 - 4ab + 6b^2$

$Z = 5a^2 - 5ab - 3b^2$

$W = 6a^2 - 4ab + 4b^2$

6.4. Symmetric equation

Ramanujan considered the Diophantine equation $x^y = y^x$. One can observe the symmetry of this equation in x and y. Ramanujan proved that this equation has only one integral equation i.e., $x = 4$ and $y = 2$ and there are an infinite number of rational solutions. For example, one has $(\frac{27}{8})^{\frac{9}{4}} = (\frac{9}{4})^{\frac{27}{8}}$.

6.5. Ramanujan's equation

The Diophantine equation $x^2 + 7 = 2^n$ is called Ramanujan's equation. He gave the solutions $1^2 + 7 = 2^3$, $3^2 + 7 = 2^4$, $5^2 + 7 = 2^5$, $11^2 + 7 = 2^7$, $181^2 + 7 = 2^{15}$ and conjectured that the above equation has no other integral solution [11]. This equation remained unsolved till 1948. Working in the quadratic number field $Q(\sqrt{-7})$, T. Nagell [8] proved Ramanujan's conjecture in 1948. He publisehd the English version of this paper in [9].

Ramanujan's equation has a bearing on triangular numbers and Mersenne numbers. A triangular number has the form $\frac{m(m+1)}{2}$. Numbers of the form $2^k - 1$ are known as Mersenne numbers. Let us consider the question : Which triangular numbers are also Mersenne numbers? Suppose $\frac{m(m+1)}{2} = 2^k - 1$ for some m and k. Then $m^2 + m + 2 = 2^{k+1}$. This can be rewritten as $4m^2 + 4m + 1 = 2^{k+3} - 7$. Hence we have $(2m+1)^2 + 7 = 2^{k+3}$. Thus we are led to Ramanujan's equation.

The utility of Ramanujan's equation

Ramanujan's equation finds applications in coding theory. One may refer to the papers by H. S. Shapiro and L. D. Slotnick [16], R. Alter [1] and E. L. Cohen [3]. Several results pertaining to the generalization of Ramanujan's equation have been described by E. L. Cohen [4] and the author [15].

7. Concluding remarks

For a survey of Ramanujan's works, one may refer to B. C. Berndt, Y. S. Choi and S. Y. Kang [2]. The collected papers of Ramanujan have been published in [12],

[13] and [14].

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3	BS-16-074	Subhashree ojha.
4	BS-16-145	Smruti Samantaray.
5	BS-16-138	Bisakhee Rout.
6	BS-16-159	Suchitra Sahoo
7	BS-16-136	Harapriya Ghadei
8	BS-16-078	Soumarekha Sahoo
9	BS-18-139	Manisha Swain
10	BS-18-138	Elina Swain
11	BS-18-080	Usharani Mohanty
12	BS-18-026	Ashruta Bhuyan
13	BS-18-128	saswatika Rath.
14	BS-18-004	Anyasha Panda
15	BS-18-064	Sitashree Mohanta.
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19	BS-17-104	Swagatika Das.
20	BS-17-054	sagarika Nayak
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22	BS-17-068	Swaraj Kumar Deep
23	BS-16-158	TApeswar Pradhan
24	BS-16-046	chandrabhanu patra
25	BS-16-020	Ashoka kumar sethi
26	BS-16-051	Suryaranjan sahoo
27	BS-17-074	[illegible]
28	BS-16-057	Sunil sethi
29	BS-16-096	Chudra sekhar Rout
30	BS-16-125	BISWOJIT pradhan.
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32	BS-16-070	Malaya RanJan Swain
33	BS-18-105	Yamunakanta Roy

34	BS-17-135	Ansuman Sahoo
35	BS-18-091	Satya Narayan Das
36	BS-18-075	Kailash Chandra Malik
37	BS-18-045	Raghabendra Biswal
38	BS-18-107	Satyajit Mallick
39	BS-17-158	Debabrata Khatua
40	BS-17-039	Nandakishore Rout
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